EFFECTIVE ELASTIC PROPERTIES OF ALUMINA-ZIRCONIA COMPOSITE CERAMICS - PART 1. RATIONAL CONTINUUM THEORY OF LINEAR ELASTICITY

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Submitted November 26, 2002; accepted January 20, 2003

Keywords: Alumina-zirconia composite ceramics, Effective elastic properties, Linear elasticity, Rational mechanics.

In this first paper of a series on the effective elastic properties of alumina-zirconia composite ceramics the theoretical framework in which these properties arise, the linear theory of elasticity, is presented in an unconventional way. A rational continuum approach is chosen, but without the formal details necessary for a mathematically strict formulation. Using a referential (Lagrangian) formulation as long as useful, the constitutive equation for the stress tensor is derived for isotropic as well as for anisotropic materials. Particular emphasis is laid on the distinction between the (geometrical) linearization of the kinematic measures (strain tensors) and the (physical) linearization of the constitutive equation (material model). Recent results occurring in the literature are mentioned. Some standard textbook formulae are recalled for the purpose of easy reference in the subsequent papers of this series.

INTRODUCTION

Due to their brittleness, the majority of advanced ceramics, and most ceramic materials in general, exhibit linear elastic behavior up to a certain critical value of strain or stress. When this critical threshhold value is exceeded, the material usually reacts to external loading by brittle fracture, long before non-linear elastic or plastic behavior has a chance to occur. Similarly, when the temperature is changed from room temperature to higher temperatures, ceramic materials usually expand by not more than a few percent. These and other general features of material behavior justify a linear treatment of the mechanics and thermomechanics of ceramic materials.

The importance of the linear theory of elasticity and thermoelasticity consists not only in its intrinsic simplicity, but also in the fact that the coefficients occuring in the linear constitutive equations (material models) are traditionally called material properties. Knowledge of these material properties, which can be measured by standard test procedures, should be sufficient to characterize material behavior in any geometrical situation.

In this first paper the linear theory of elasticity is presented in a modern way, using a rational continuum approach. In order to make this introductory paper as concise as possible and in order to make it easily accessible to a wide audience of readers, however, the details necessary for a mathematically strict formulation are discarded in all cases where no essential additional physical insight is gained by them. According to the didactic character of this introductory paper emphasis is laid on the final formulae and their physical meaning. For general reference, a mathematically more stringent formulation, comprehensive references to original work and further reading the reader should refer to [1-3]. In this paper, formal details are to a large degree replaced by intuitive arguments. In order not to complicate notation, only absolutely indispensible expressions are given explicitly. Relations (however important they may be) that are not needed for further reference in this series of papers are mentioned or explained only in the text or in footnotes wherever possible. The purpose of this paper is to give the reader a red line for a thorough understanding of linear elasticity theory that is indispensible for an advanced treatment of the effective elastic properties of alumina-zirconia composite ceramics. With respect to this specific purpose, many analogous or similar relations that are usually treated in modern textbooks of elasticity theory are omitted. When need arises, the interested reader may consult e.g. [4-6].

Deformation and strain tensors

Mechanics in general deals with motion. Continuum mechanics deals with motion of material bodies, consisting of so-called material particles. The most fundamental function describing motion of material bodies in continuum mechanics is the so-called deformation function (transplacement) $\boldsymbol{\chi}$. This vector function uniquely determines the actual position **x** of a material particle (in the sense of continuum mechanics) at time *t*, which has been at a referential position **X** (in an arbitrarily chosen reference frame) at an arbitrarily chosen reference time:

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X},t) \tag{1}$$

For fixed **X** (i.e. selecting a certain material particle) $\boldsymbol{\chi}$ determines a trajectory, for fixed *t* (i.e. imaging the whole body at a certain moment) $\boldsymbol{\chi}$ determines a configuration (placement). The referential gradient (i.e. the derivative with respect to the referential position) of the deformation function is a second-order tensor, the so-called deformation gradient (transplacement gradient)

$$\mathbf{F} \equiv \mathrm{Grad}\,\mathbf{x} = \frac{\partial\,\boldsymbol{\chi}(\mathbf{X},t)}{\partial \mathbf{X}} \tag{2}$$

While the function $\boldsymbol{\chi}$ contains complete information about the motion of a material body (translation, rotation, and change of shape), the deformation gradient **F** is a local measure of motion and contains information only on rotation and change of shape. Materials that can be modelled by using the deformation gradient (2) only (i.e. without invoking deformation gradients of higher order) are called simple materials (materials with local response, i.e. materials for which the principle of local action can be adopted). Materials for which (2) is not a sufficient measure of deformation, are called non-local. Elasticity theory (linear and non-linear) deals only with simple materials.

For the purpose of linear elasticity it is useful to introduce another vector function, the so-called displacement, via the definition

$$\mathbf{u}(\mathbf{X},t) \equiv \boldsymbol{\chi}(\mathbf{X},t) - \mathbf{X} \tag{3}$$

Using equation (3), the deformation gradient can be written as

$$\mathbf{F} \equiv \mathrm{Grad}\,\mathbf{x} = \frac{\partial\,\boldsymbol{\chi}(\mathbf{X},t)}{\partial \mathbf{X}} = \frac{\partial\,[\mathbf{X} + \mathbf{u}(\mathbf{X},t)]}{\partial \mathbf{X}} = \mathbf{1} + \mathrm{Grad}\,\mathbf{u} \tag{4}$$

where Grad **u** is the displacement gradient and **1** the second-order unit tensor. According to Cauchy's polar decomposition theorem (cf. e.g. [1-5])[1], the deformation gradient can be uniquely decomposed into a (pro-per) orthogonal tensor **R** (corresponding to rigid body rotations) and a symmetric, positively definite tensor **U**

$$\mathbf{F} = \mathbf{R}\mathbf{U} \tag{5}$$

where **R** and **U** have the properties

$$\mathbf{R}\mathbf{R}^T = \mathbf{1} \tag{6}$$

$$\det \mathbf{R} = +1 \tag{7}$$

$$\mathbf{U} = \mathbf{U}^T \tag{8}$$

$$(\mathbf{U}\mathbf{a})\,\mathbf{a} > 0 \tag{9}$$

where equation (9) holds for arbitrary vectors $\mathbf{a} \neq 0$ and the superscript *T* denotes transposition. The second-order tensor **U** is called right stretch tensor[2]. Note that the deformation gradient **F** is not symmetric, in contrast to **U**. Based on **U**, other (non-linear) deformation measures can be defined. In particular, in order to avoid the irrational operation of taking the square root of a certain product of two tensors (which would require definition)

$$\mathbf{U} = \sqrt{\mathbf{F}^T\,\mathbf{F}} \tag{10}$$

one usually prefers the so-called right Cauchy-Green deformation tensor, defined as

$$\mathbf{C} \equiv \mathbf{U}^2 = \mathbf{F}^T\,\mathbf{F} \tag{11}$$

For rigid body motions (and for the reference configuration as a special case) the deformation tensors reduce to the unit tensor **1**. That means that in the absence of deformations (i.e. in the absence of changes of shape of a body) the unit tensor results. For many practical purposes it is more useful, however, to describe the absence of deformations by the null tensor (zero tensor) **0**. This is the impetus for defining so-called strain measures, in addition to deformation measures. The most popular (non-linear) strain measure is the Green-Lagrange strain tensor

$$\mathbf{G} \equiv \tfrac{1}{2}\,(\mathbf{C} - \mathbf{1}) = \tfrac{1}{2}\,(\mathrm{Grad}\,\mathbf{u} + \mathrm{Grad}\,\mathbf{u}^T + \mathrm{Grad}\,\mathbf{u}^T\,\mathrm{Grad}\,\mathbf{u}) \tag{12}$$

Note that both **C** and **G** are symmetric tensors of second order. They are deformation and strain measures valid for finite (i.e. arbitrarily large) deformations and strains.

[1] Cauchy's polar decomposition theorem is valid for regular (i.e. non-singular) tensors, i.e. tensors with a non-zero jacobian. Regularity is a consequence of the fact that the deformation function is invertible, which means in physical terms that neither crossing, nor creation, nor destruction of trajectories is allowed.

[2] A completely analogous formulation is possible by using the left stretch tensor **V** and the corresponding counterpart form of Cauchy's polar decomposition theorem **F**=**VR**. Since no essential physical insight is gained by making this counterpart formulation explicit, it is omitted here. When needed, the reader can easily find the respective relations in standard textbooks and monographs of rational continuum mechanics, cf. [1-5].

Stress tensors

The most important law of continuum mechanics is the momentum balance. In words, it states that the material time derivative of momentum (i.e. in the non-relativistic case the product of mass and acceleration) is balanced by the sum of forces. In continuum mechanics it turned out useful to distinguish between forces of two different ranges, long-range forces acting on the whole volume of the body (body forces) and short-range forces acting on short distances. In classical continuum mechanics the short-range forces are assumed to be effectively cancelled everywhere inside the body as a consequence of the law of action and reaction. Note that this need not be the case for (rational) mixture theories[3]. At the external boundary (i.e. the surface) of the body, however, the short-range forces will act (on the surroundings), since there is no counterforce to balance their effect. These surface forces are usually described by a stress vector (traction vector), which depends on the position **x**, the time t and the surface on which it acts, oriented with a direction described by the normal vector **n** of the (hypothetical or actual) surface to which it is perpendicular:

$$\mathbf{t} = \mathbf{t}\,(\mathbf{x}, t, \mathbf{n}) \tag{13}$$

Thus, the stress vector is not a field in the classical sense. Fortunately, it turns out (as can be demonstrated by Cauchy's classcial tetrahedron argumentation, cf. e.g. [1-7]), that the dependence on **n** is a linear one:

$$\mathbf{t} = \mathbf{T}\,\mathbf{n} \tag{14}$$

where $\mathbf{T} = \mathbf{T}(\mathbf{x},t)$ is the Cauchy stress tensor, a symmetric (due to the moment of momentum balance) tensor of second order.

Considering the simplest mechanical test, uniaxial extension of a cylindrical body (a rod), the traction vector (at any time of the test) corresponds to the force acting perpendicularly to the direction of extension divided by the actual cross-section area of the test rod (corresponding to the time instant at which the force is measured). In practice, however, it is often more convenient (and absolutely sufficient) to measure the cross-section area once and for all at the beginning of the test, instead of recording it simultaneously with the force measurement during the whole test. In this case the cross-section area is a referential one and, correspondingly, also the stress vector. Based on this (referential or nominal) stress vector, several referential stress tensors can be defined, cf. e.g. [1-4]. For the purpose of demonstrating how the linear theory of elasticity emerges from the non-linear theory the second Piola-Kirchhoff stress tensor is the most convenient. This symmetric second-order tensor is connected to the Cauchy stress tensor via the relation

$$\mathbf{S} = |\det \mathbf{F}|\, \mathbf{F}^{-1}\, \mathbf{T}(\mathbf{F}^{-1})^T \tag{15}$$

The most general material model for elastic solids states that the second Piola-Kirchhoff stress tensor is a function of the Green-Lagrange strain tensor. Thus, the most general constitutive equation of elastic solids is

$$\mathbf{S} = \mathbf{S}\,(\mathbf{G}) \tag{16}$$

This abstract material model is general enough to describe the behavior of anisotropic non-linearly elastic solids in finite deformation processes. In order to make the constitutive equation more explicit, however, additional specifying assumptions have to be adopted concerning material behavior.

Physical linearization of the constitutive equation of isotropic solids

For isotropic solids the constitutive equation, equation (16), can readily be written in a more explicit form without loss of generality. Invoking the Cayley-Hamilton theorem and the representation theorem for isotropic tensor functions with symmetric tensor arguments one obtains the formally simple expression, cf. [1-4]

$$\mathbf{S} = \gamma_0\,(...)\,\mathbf{1} + \gamma_1\,(...)\,\mathbf{G} + \gamma_2\,(...)\,\mathbf{G}^2 \tag{17}$$

where (...) indicates that the scalar coefficients γ_0, γ_1, γ_2 are functions of three scalar invariants of **G**, e.g. tr**G**, tr($\mathbf{G}^2$) and tr($\mathbf{G}^3$). Obviously, equation (17) can be linearized by avoiding occurrence of non-linear terms in **G**, i.e. omitting the third r.h.s. term, making γ_1 a constant and permitting only a linear dependence of γ_0 on tr**G**. The resulting linear constitutive equation for isotropic elastic solids is:

$$\mathbf{S} = \gamma_0 \cdot (\mathrm{tr}\mathbf{G}) \cdot \mathbf{1} + \gamma_1\,\mathbf{G} \tag{18}$$

Note that at this stage no assumption has been made concerning the magnitude of the deformations (strains). These can be arbitrarily large (finite). Note also that residual stresses (corresponding, say, to a S_0 term on the r.h.s.) are usually neglected, based on the argumentation that the referential state is chosen to be stress-free. For anisotropic solids, equation (16) could directlly be linearized by assuming the relation

$$\mathbf{S} = \mathbf{K}\mathbf{G} \tag{19}$$

where **K** is a fourth-order tensor. In order to preserve maximum accordance with textbook tradition, however,

[3] In rational mixture theories short-range forces between the constituents of the mixture (inside the mixture body) are related to momentum exchange or interaction forces, cf. e.g. [8-10].

we will postpone the linearization for anisotropic solids (and its subsequent specification to selected symmetries, including isotropy) to a later stage where the geometrical linearization of the kinematic measures has already been introduced.

Geometrical linearization of the kinematic measures

Small (infinitesimal) deformations (strains) are defined by the fact that the displacement gradients are small. Since displacement gradients Grad **u** are tensors, their smallness can only be evaluated when an appropriate scalar measure (norm, magnitude) of these tensors is defined which can be compared to unity. With this scalar measure the condition for small strains can be formulated as follows:

$$\sqrt{\text{Grad } \mathbf{u} \cdot (\text{Grad } \mathbf{u})^T} << 1 \tag{20}$$

Some of the consequences of condition (20) are

$$\mathbf{F} \approx 1 \tag{21 a}$$

$$|\det \mathbf{F}| \approx 1 \tag{21 b}$$

$$\mathbf{S} \approx \mathbf{T} \tag{21 c}$$

and the referential gradient (Grad) reduces to the usual spatial gradient (grad.). In other words, in the case of small strains there is no reason to distinguish between referential (Lagrangian) and spatial (Eulerian) description. This is assumed in many undergraduate texts on elasticity theory. In particular, when one confines oneself a priori to small strains, the second Piola-Kirchhoff stress tensor reduces to the Cauchy stress tensor and thus only one stress tensor has to be introduced from the beginning.

In the case of small strains the (non-linear) Green-Lagrange strain tensor (12) reduces to the so-called small strain tensor, defined as

$$\mathbf{E} \equiv \tfrac{1}{2} (\text{grad } \mathbf{u} + \text{grad } \mathbf{u}^T) \tag{22}$$

Similar to **G**, the small strain tensor **E** is a symmetric second-order tensor. Note, however, that, in contrast to **G**, **E** is a linear strain measure, i.e. suitable only for small deformations.

Anisotropic solids

Specifying equation (19) to the case of small strains (i.e. invoking physical and geometrical linearization) yields Hooke's law for anisotropic solids in direct tensor notation

$$\mathbf{T} = \mathbf{C}\,\mathbf{E} \tag{23}$$

where **C** is the elasticity tensor, a fully symmetric fourth-order tensor. Using index tensor notation, Hooke's law can be written in the form

$$T_{ij} = C_{ijkl}\, E_{kl} \tag{24}$$

As a consequence of the symmetry of **T** and **E**, the number of components of this tensor is reduced from a total of 81 to 36 independent ones [1, 2, 4, 7]. Thus, it is possible to represent this fourth-order tensor alternatively in the form of a (6×6) matrix (which, of course, does not have the transformation properties of a tensor), and to express Hooke's law in the direct matrix notation (engineering notation) [4, 6, 7]

$$[\sigma] = [C]\,[\varepsilon] \tag{25}$$

or, in index matrix notation,

$$\sigma_i = C_{ij}\, \varepsilon_j \tag{26}$$

In this notation $[\sigma]$ and $[\varepsilon]$ are 6-dimensional column vectors referring to stress and strain, respectively, the components of which are defined as follows

$$\sigma_1 \equiv T_{11}, \sigma_2 \equiv T_{22}, \sigma_3 \equiv T_{33}, \sigma_4 \equiv T_{23}, \sigma_5 \equiv T_{31}, \sigma_6 \equiv T_{12} \tag{27}$$

$$\varepsilon_1 \equiv E_{11}, \varepsilon_2 \equiv E_{22}, \varepsilon_3 \equiv E_{33}, \varepsilon_4 \equiv E_{23}, \varepsilon_5 \equiv E_{31}, \varepsilon_6 \equiv E_{12} \tag{28}$$

and $[C]$ is the so-called stiffness matrix, the elements of which are given by the substitution of index pairs as follows:

$$11 \rightarrow 1,\ 22 \rightarrow 2,\ 33 \rightarrow 3,\ 23 \rightarrow 4,\ 31 \rightarrow 5,\ 12 \rightarrow 6$$

Additionally it is usually assumed[4] that the stiffness itself is symmetric with respect to its diagonal, i.e. the total number of independent components is reduced from 36 to 21 (so-called Green elasticity or hyperelasticity[5]). Thus in the most general case of well-defined anisotropy (triclinic monocrystals) the (6x6) stiffness matrix or, alternatively, the fourth-order elasticity tensor, has 36 elastic constants (elastic moduli), 21 of which can be assumed to be independent. In matrix index notation:

$$[C_{triclinic}] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \tag{29}$$

[4] This assumption is based on a more fundamental assumption concerning the elastic energy (stored energy function) [1, 2, 7, 11]. If the elastic energy, which is a potential function for the stress tensor, vanishes in the unstrained state and can be expressed by a symmetric quadratic form, then the stiffness matrix is symmetric, i.e. the elasticity tensor is fully symmetric.

[5] In contrast to the so-called Cauchy elasticity where this is not the case.

For the off-diagonal elements in this stiffness matrix and the following ones we automatically assume validity of the symmetry condition for Green elasticity (hyperelasticity):

$$C_{ij} = C_{ji} \tag{30}$$

For materials of higher symmetry (monocrystals or polycrystalline bodies) the number of independent elastic constants is further reduced as follows:

- Monoclinic monocrystals (13 independent elastic constants):

$$[C_{monoclinic}] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{21} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{31} & C_{32} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & C_{46} \\ 0 & 0 & 0 & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & 0 & 0 & C_{66} \end{bmatrix} \tag{31}$$

- Orthorhomic monocrystals and orthotropic polycrystalline bodies (9 independent elastic constants):

$$[C_{orthorombic}] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \tag{32}$$

- Trigonal monocrystals (6 independent elastic constants):

$$[C_{trigonal}] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ C_{21} & C_{22} & C_{23} & C_{24} & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0 \\ C_{41} & C_{42} & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & C_{56} \\ 0 & 0 & 0 & 0 & C_{65} & C_{66} \end{bmatrix} \tag{33}$$

with the additional conditions

$$C_{11} = C_{22},\ C_{13} = C_{23},\ C_{44} = C_{55} \tag{34}$$

and

$$C_{14} = -C_{24} = C_{56},\ C_{66} = \tfrac{1}{2}\,(C_{11} - C_{12}) \tag{35}$$

For tetragonal monocrystals (6 independent elastic constants)[6] the general form of the orthorhombic (orthotropic) stiffness matrix, equation (32), together with conditions (30) and (34) applies. The same holds true for materials with higher than tetragonal symmetry. In these cases, however, the following additional conditions hold for the non-zero elements:

- Hexagonal monocrystals and transversely isotropic polycrystalline bodies (5 independent elastic constants):

$$C_{66} = \tfrac{1}{2}\,(C_{11} - C_{12}) \tag{36}$$

- Cubic monocrystals (3 independent elastic constants):

$$C_{11} = C_{22} = C_{33},\ C_{12} = C_{23} = C_{31},\ C_{44} = C_{55} = C_{66} \tag{37}$$

- Isotropic materials (2 independent elastic constants):

$$C_{11} = C_{22} = C_{33},\ C_{12} = C_{23} = C_{31},\ C_{44} = C_{55} = C_{66} = \tfrac{1}{2}\,(C_{11} - C_{12}) \tag{38}$$

In the case of isotropic materials the stiffness matrix can be written as follows (using the definitions and $C_{12} \equiv \lambda$ and $C_{44} \equiv \mu$):

$$[C_{isotropic}] = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \tag{39}$$

The elastic constants (elastic moduli) λ and μ are called Lamé constants (units [GPa]). Switching over from matrix notation to tensor notation the elasticity tensor is

$$C_{ijkl} = \lambda\ \delta_{ij}\ \delta_{kl} + \mu\ (\delta_{ik}\ \delta_{jl} + \delta_{il}\ \delta_{jk}) \tag{40}$$

where the δ's are Kronecker deltas. Inserting the elasticity tensor given by equation (40) into Hooke's law, equation (24), yields the Cauchy-Hooke law for isotropic materials:

$$\begin{aligned} T_{ij} &= [\lambda\ \delta_{ij}\ \delta_{kl} + \mu\ (\delta_{ik}\ \delta_{jl} + \delta_{il}\ \delta_{jk})] \\ E_{kl} &= \lambda\ \delta_{ij}\ \delta_{kl}\ E_{kl} + 2\mu\ E_{ij} = \lambda\ \delta_{ij}\ E_{kk} + 2\mu\ E_{ij} \end{aligned} \tag{41}$$

Switching over from index tensor notation to direct tensor notation this corresponds to

$$\mathbf{T} = \lambda \cdot (\mathrm{tr}\mathbf{E}) \cdot \mathbf{1} + 2\mu\ \mathbf{E} \tag{42}$$

This equation can easily be recognized as being linearized with respect to the material model (physical linearization) and the kinematic measures (geometric linearization), cf. equations (18) and (23).

[6] Most textbooks claim that in the case of trigonal and tetragonal monocrystals it is necessary to distinguish two cases of elastic symmetry, depending on the crystal class (point group): one with 7 independent elastic constants (point groups 3, /3 4, /4, 4/*m*) and one with 6 independent elastic constants (point groups 32, 3*m*, /3*m*, 422, 4*mm*, /42*m*, 4/*m* 2/*m* 2/*m*). As shown in a recent, highly recognized, paper by Forte and Vianello [12], this claim is wrong.

Engineering elastic moduli of isotropic materials

The two Lamé constants occurring in equations (39)-(42) are one possible choice of elastic constants (elastic moduli) in the case of isotropic materials. Depending on the application in question, other elastic moduli can be more advantageous. E.g. the tensile modulus (Young modulus) E (units [GPa]) and the Poisson ratio ν (dimensionless) are preferable from the practical point of view, since they can be relatively easily determined by standard test procedures. The shear modulus G (units [GPa]) and bulk modulus K (units [GPa]), on the other hand, are preferable from the theoretical point of view, e.g. for micromechanical calculations.

In terms of E and ν (both measurable in principle in a uniaxial tension test) the Cauchy-Hooke law can be written as

$$\mathbf{T} = \frac{E}{(1+\nu)}\left[\mathbf{E} + \frac{\nu}{(1-2\nu)} \cdot (\mathrm{tr}\mathbf{E}) \cdot \mathbf{1}\right] \qquad (43)$$

From this equation it is evident that for the Poisson ratio ν the values 0.5 and –1 are not allowed. Actually, as a consequence of the second law of thermodynamics the following inequality must hold [2, 5, 6]:

$$-1 < \nu < 0.5 \qquad (44)$$

Although it has been known for at least 75 years that anisotropic materials (e.g. pyrite monocrystals) can exhibit negative Poisson ratios in certain directions [11], until recently most textbooks on the mechanics of materials suggested that according to experience with real materials the Poisson ratio should be positive for all isotropic materials (i.e. $0 < \nu < 0.5$), cf. e.g., very recently, [5]. This misleading suggestion was supported by the intuitive interpretation of the Poisson ratio: in the limit of incompressible materials $\nu \to 0.5$, while the case $\nu = 0$ corresponds to a (compressible) material that does not contract in uniaxial extension perpendicular to the extension direction. Research in materials science for more than a decade has clearly shown, however, that isotropic materials with negative Poisson ratio exist [13-15]. These materials show the unexpected behavior, that when extended in one direction, they extend also in all perpendicular directions. Potential applications are so-called press-fit fasteners, which are easily inserted into a hole but their removal is resisted because the fasteners expand under tension [15]. Most dense ceramic materials exhibit Poisson ratios in the range 0.1 – 0.3 [16, 17] and for many purposes the approximate value of 0.2 can serve as a reasonable estimate in the absence of more precise information (for alumina and zirconia approximately 0.25). Nevertheless, new research in ceramic science does not exclude the occurrence of negative Poisson ratios [18].

In a similar way as the Young modulus E and the Poisson ratio ν are connected to the uniaxial extension test, the shear modulus G and the bulk modulus K are connected to simple shear and isotropic deformation (i.e. dilatation or compression), cf. e.g. [2, 4-7]. Note that, accidentally, it turns out that the shear modulus equals the second Lamé constant μ.

Since, as mentioned above, two independent elastic constants (moduli) are necessary and sufficient for the complete description of the elastic response of isotropic materials, the remaining elastic constants (moduli) are not independent, i.e. any elastic constant can be expressed in terms of two other independent ones. Table I lists some useful relations between popular elastic constants of isotropic materials (note that $G = \mu$), cf. e.g. [2, 4-7].

SUMMARY AND OUTLOOK

In this first, introductory, part of a series of papers on the effective elastic properties of alumina-zirconia composite ceramics the linear theory of elasticity has been presented in a condensed form, as far as possible exact, but without superfluous mathematical details. In order to emphasize physical content at the cost of technical detail, the linear theory for small deformations has been derived as a special case of the non-linear theory with finite deformations, in contrast to comparable treatments. The authors believe that this is the way of choice mechanics of materials should be taught to materials scientists, in contrast to civil and mechanical engineers. Starting with the basic concepts of continuum mechanics (deformation function, deformation gradient, displacement gradient), non-linear deformation and strain measures are introduced (Cauchy-Green deformation tensor, Green-Lagrange strain tensor) as well as spatial and referential stress tensors (Cauchy stress ten-

Table 1. Some relations between popular elastic constants of isotropic materials.

	λ, μ	E, ν	G, ν	K, ν	G, K	E, G
E	$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	E	$2G(1+\nu)$	$3K(1+2\nu)$	$\frac{9KG}{3K+G}$	E
ν	$\frac{\lambda}{2(\lambda+\mu)}$	ν	ν	ν	$\frac{3K-2G}{2(3K+G)}$	$\frac{E-2G}{2G}$
G	μ	$\frac{E}{2(1+\nu)}$	G	$\frac{3K(1-2\nu)}{2(1+\nu)}$	G	G
K	$\frac{3\lambda+2\mu}{3}$	$\frac{E}{3(1-2\nu)}$	$\frac{2G(1+\nu)}{3(1-2\nu)}$	K	K	$\frac{EG}{3(3G-E)}$

sor, second Piola-Kirchhoff stress tensor). Emphasizing the didactic character of this paper, linearization is introduced in two clearly distinguished steps: first, as the physical linearization of the constitutive equation (via Cayley-Hamilton theorem and representation theorem for isotropic tensor functions) and second as the geometrical linearization of the kinematic measures (leading to the definition of the small strain tensor). Anisotropic materials are treated with physical and geometrical linearization. The elasticity tensors or stiffness matrices for materials with different symmetry are given and discussed. It is mentioned that, in contrast to common belief and textbook tradition, recently published work shows that trigonal and tetragonal monocrystals with 7 independent elastic constant cannot exist. Apart from the Lamé constants, engineering elastic constants and their mutual relationsships in the case of isotropic materials are mentioned and briefly discussed (Young modulus, Poisson ratio, shear modulus, bulk modulus). In particular, it is mentioned that, in contrast to ceramic textbook tradition, published research since more than a decade in other fields of materials science has clearly shown that isotropic materials with negative Poisson ratio exist and are already going to be applied.

The present paper will serve as a theroetical basis and handy reference for the exact treatment of the effective elastic properties of alumina-zirconia composite ceramics in the subsequent papers of this series. These papers will concern the micromechnical approach to porous ceramics and two-phase ceramic composites in general, the calculation of effective elastic moduli of dense polycrystalline alumina and zirconia from elastic moduli of alumina and zirconia monocrystals, the prediction of effective elastic moduli of dense alumina-zirconia composites, as well as effective elasticity moduli of porous ceramics in the alumina-zirconia system.

Acknowledgement

This work was part of the project "Mechanics and Thermomechanics of Disperse Systems, Porous Materials and Composites", supported by the Grant Agency of the Czech Republic (Grant No. 106/00/D086).

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EFEKTIVNÍ ELASTICKÉ VLASTNOSTI KOMPOZITNÍ KERAMIKY NA BÁZI Al_2O_3 A ZrO_2 - ČÁST 1. LINEÁRNÍ ELASTICITA V RÁMCI RACIONÁLNÍ TEORIE KONTINUA

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V tomto prvním článku, na který by měla navazovat řada prací zabývajících se efektivními elastickými vlastnostmi kompozitní keramiky na bázi Al_2O_3 a ZrO_2, je vytyčen teoretický rámec, ve kterém jsou tyto vlastnosti definovány. Lineární teorie elasticity je zde prezentována v rámci racionální teorie kontinua, ale bez formálních detailů, které by pro tento účel zbytečně zatížily matematický popis. Oproti konvenčnímu přístupu je zde lineární teorie elasticity pro případ malých deformací odvozena jako speciální případ nelineární teorie pro velké deformace. Zvláštní důraz je kladen na rozlišení fyzikální linearizace konstitutivní rovnice a geometrické linearizace kinematických veličin. Elastické vlastnosti anizotropních materiálů jsou prezentovány až po zavedení fyzikální a geometrické linearizace. Jsou zde uvedeny matice tuhosti pro materiály všech symetrií (monokrystalů a polykrystalických keramik resp. kompozitů). V rámci jejich diskuse je citován relativně nový poznatek z roku 1996 z práce Forte a Vianella: nemožnost existence trigonálních a tetragonálních monokrystalů se 7 nezávislými elastickými konstantami. Kromě Laméových konstant jsou uvedeny a oddiskutovány tzv. „inženýrské“ elastické konstanty (Youngův modul, Poissonův poměr, smykový modul a modul stlačitelnosti) a jejich vzájemné vztahy pro případ izotropních materiálů. Je zde presentován fakt, který je dosud často opomíjen v základních učebnicích (především v oblasti keramiky): totiž, že existují materiály (dokonce izotropní) s negativním Poissonovým poměrem.